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Canonical and D-Transformations in Theories with Constraints

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We describe a class of transformations in a super phase space (we call them D-transformations) which play the role of ordinary canonical transformations in theories with second-class constraints. Namely, in such theories they preserve the form invariance of equations of motion, their quantum analogs are unitary transformations, and the measure of integration in the corresponding Hamiltonian path integral is invariant under these transformations.

1. INTRODUCTION

Canonical transformations play an important role in the Hamiltonian formulation of classical mechanics without constraints (Landau and Lifshitz, 1973). They preserve the form invariance of the Hamiltonian equations of motion and their quantum analogs are unitary transformations (Dirac, 1958; Weyl, 1950). Canonical transformations constitute also a powerful tool of classical mechanics, which allow one often to simplify solutions of the theory. For example, it is enough to mention that evolution is also a canonical transformation. Quantum implementation of canonical transformations have been discussed in numerous papers (e.g., DeWitt, 1951; Itzykson, 1967; Moshinsky and Quesne, 1971; Anderson, 1994). However, modern physical theories in their classical versions are mostly singular (in particular, gauge) ones, which means that in the Hamiltonian formulation they are theories with constraints (Dirac, 1964; Gitman and Tyutin, 1986, 1990). Equations of a Hamiltonian theory with constraints are not form invariant under canonical transformations, but this circumstance allows one to use these transformations to simplify the equations and to clarify the structure of the gauge theory in the Hamiltonian formulation. Moreover, formulations of a gauge theory in

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two different gauges are connected by means of a canonical transformation (Gitman and Tyutin, 1983, 1986, 1987, 1990). In the general case, equations of constraints change their form under the canonical transformations. This is an indirect indication that the quantum version of the canonical transformations in constrained theories is not a unitary transformation (of course, we are speaking about the complete theory, but not about its reduced unconstrained version). Thus, in the case of constrained theories one can believe that besides the canonical transformation, another kind of transformation has to exist, which preserves the form invariance of the equations of motion and which induces unitary transformations on the quantum level. Namely, they play the role of ordinary canonical transformations in theories without constraints.

In this paper we describe such a kind of transformation for theories with second-class constraints, which is, in fact, a general case, because a theory with first-class constraints can be reduced to a theory with second-class constraints by a gauge fixing. We call such transformations D-transformations.

2. GENERALIZED CANONICAL TRANSFORMATIONS

Let a classical mechanics be given with phase variables $\eta = (\eta^A)$, $A = 1, \ldots, 2n$ [in the general case they belong to Berezin algebra (Berezin, 1965, 1983, 1987; Gitman and Tyutin, 1986, 1990) and have the Grassmann parities $P(\eta^A) = P_A$] and with a symplectic metrics $\Lambda^{AB}(\eta)$, which defines a generalized super Poisson bracket for any two functions $F(\eta)$ and $G(\eta)$ with definite Grassmann parities $P(F)$ and $P(G)$,

$$\{F, G\}^{(\eta,\Lambda)} = \frac{\partial_r F}{\partial\eta^A} \Lambda^{AB}(\eta) \frac{\partial_l G}{\partial\eta^B} \tag{2.1}$$

where $\partial_r/\partial\eta^A$ and $\partial_l/\partial\eta^B$ are the right and left derivatives, respectively. The metrics $\Lambda^{AB}(\eta)$ is a T_2-antisymmetric supermatrix (Gitman and Tyutin, 1986, 1990), $P(\Lambda^{AB}) = P_A + P_B$, $\Lambda^{AB}(\eta) = -(-1)^{P_A P_B}\Lambda^{BA}(\eta)$, obeying the condition

$$(-1)^{P(A)P(C)}\Lambda^{AD}(\eta) \frac{\partial_l \Lambda^{BC}(\eta)}{\partial\eta^D} + \text{cycl}(A, B, C) = 0 \tag{2.2}$$

which is necessary and sufficient for the bracket (2.1) to be super antisymmetric and satisfy the super Jacobi identity,

$$\{F, G\}^{(\eta,\Lambda)} = -(-1)^{P(F)P(G)}\{G, F\}^{(\eta,\Lambda)}$$

$$(-1)^{P(F)P(K)}\{\{F, G\}^{(\eta,\Lambda)}, K\}^{(\eta,\Lambda)} + \text{cycl}(F, G, K) = 0 \tag{2.3}$$

In addition, the following property holds:

$$\{F, GK\}^{(\eta,\Lambda)} = \{F, G\}^{(\eta,\Lambda)}K + (-1)^{P(F)P(G)}G\{F, K\}^{(\eta,\Lambda)} \tag{2.4}$$

It is easy to see that

$$\Lambda^{AB}(\eta) = \{\eta^A, \eta^B\}^{(\eta,\Lambda)} \tag{2.5}$$

In the case

$$\Lambda^{AB} = E^{AB} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

the generalized Poisson bracket (2.1) coincides with the ordinary super Poisson bracket,

$$\{F, G\}^{(\eta,E)} = \frac{\partial_r F}{\partial \eta^A} E^{AB} \frac{\partial_l G}{\partial \eta^B} = \{F, G\} \tag{2.6}$$

If $\eta' = \eta'(\eta)$ is a nonsingular change of variables, then the generalized Poisson bracket (2.1) acquires in the primed variables the form

$$\{F, G\}^{(\eta,\Lambda)} = \{F', G'\}^{(\eta',\Lambda')} = \frac{\partial_r F'}{\partial \eta'^A} \Lambda'^{AB}(\eta') \frac{\partial_l G'}{\partial \eta'^B} \tag{2.7}$$

where

$$F'(\eta') = F(\eta), \qquad G'(\eta') = G(\eta)$$

$$\Lambda'^{AB}(\eta') = \frac{\partial_r \eta'^A}{\partial \eta^C} \Lambda^{CD}(\eta) \frac{\partial_l \eta'^B}{\partial \eta^D} = \{\eta'^A, \eta'^B\}^{(\eta,\Lambda)} \tag{2.8}$$

By analogy with the case of the ordinary Poisson bracket one can ask which kind of transformation keeps the generalized Poisson bracket form invariant, namely when the following relation holds:

$$\Lambda'^{AB}(\eta') = \Lambda^{AB}(\eta') \tag{2.9}$$

We will call such transformations generalized canonical transformations. They are just canonical transformations in the case when the generalized Poisson bracket coincides with the ordinary Poisson bracket.

Consider transformations of the form

$$\eta' = [\exp(\check{W})]\eta \tag{2.10}$$

where the operator $\check{W}$ is defined by its action on functions of η,

$$\check{W}F(\eta) = \{F, W\}^{(\eta,\Lambda)} \tag{2.11}$$

where $W(\eta)$ $[P(W) = 0]$ is a generating function of the transformation. We are going to demonstrate that transformations of the form (2.10) are just the generalized canonical transformations connected continuously with the identical transformation. To this end one has first to verify that the following property holds:

$$e^{\check{W}}F(\eta) = F(e^{\check{W}}\eta) = F(\eta') \tag{2.12}$$

Indeed, one can see, using (2.4), that

$$e^{\check{W}}F(\eta)e^{-\check{W}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\check{W}, [\check{W}, \ldots, [\check{W}, F]\cdots]] = e^{\check{W}}F(\eta) \tag{2.13}$$

Then, one can write, for example, for any analytic function $F(\eta)$,

$$e^{\check{W}}F(\eta) = e^{\check{W}}F(\eta)e^{-\check{W}} = F(e^{\check{W}}\eta e^{-\check{W}}) = F(e^{\check{W}}\eta) = F(\eta')$$

Now, let us introduce a function $F^{AB}(\alpha, \eta)$, $P(\alpha) = 0$,

$$F^{AB}(\alpha, \eta) = \{e^{\alpha\check{W}}\eta^A, e^{\alpha\check{W}}\eta^B\}^{(\eta,\Lambda)} \tag{2.14}$$

At $\alpha = 0$ this function coincides with $\Lambda^{AB}(\eta)$ [see (2.5)] and at $\alpha = 1$ with $\Lambda'^{AB}(\eta')$ [see (2.8) and (2.10)],

$$F^{AB}(0, \eta) = \Lambda^{AB}(\eta) \tag{2.15}$$

$$F^{AB}(1, \eta) = \Lambda'^{AB}(\eta') \tag{2.16}$$

Differentiating (2.14) with respect to α and using the Jacoby identity (2.3), one can get an equation for the function $F^{AB}(\alpha, \eta)$,

$$\frac{\partial F^{AB}(\alpha, \eta)}{\partial \alpha} = \check{W}F^{AB}(\alpha, \eta) \tag{2.17}$$

A solution of this equation which obeys the initial condition (2.15) has the form

$$F^{AB}(\alpha, \eta) = e^{\alpha\check{W}}\Lambda^{AB}(\eta) \tag{2.18}$$

Taking into account equation (2.16) and the property (2.12), we get just the condition (2.9) of the form invariance of the generalized Poisson bracket. Thus, the transformations (2.10) are generalized canonical transformations connected continuously with the identical transformation. By definition they preserve the form invariance of the generalized Poisson bracket,

$$\{F, G\}^{(\eta,\Lambda)} = \{F', G'\}^{(\eta',\Lambda)}, \qquad F'(\eta') = F(\eta), \quad G'(\eta') = G(\eta) \tag{2.19}$$

In particular, the infinitesimal form of the generalized canonical transformations is

$$\eta' = \eta + \delta\eta, \qquad \delta\eta = \{\eta, \delta W\}^{(\eta,\Lambda)} \tag{2.20}$$

Let us suppose now that the classical mechanics in question has equations of motion of the form

$$\dot{\eta} = \{\eta, H\}^{(\eta,\Lambda)} \tag{2.21}$$

i.e., the Hamiltonian equations of motion, but with a generalized Poisson bracket. How are they transformed under the generalized canonical transformations (2.20)? The result is

$$\dot{\eta}' = \{\eta', H'\}^{(\eta',\Lambda)}, \qquad H'(\eta') = H(\eta) + \frac{\partial \delta W}{\partial t} \tag{2.22}$$

This means that equation (2.21) is form invariant under the generalized canonical transformations; only the Hamiltonian is changed, similar to the usual case of the canonical transformations and Hamiltonian equations of motion with the ordinary Poisson bracket. To see this, one has to calculate the time derivative of η', using (2.21),

$$\dot{\eta}' = \{\eta + \delta\eta, H\}^{(\eta,\Lambda)} + \left\{\eta, \frac{\partial \delta W}{\partial t}\right\}^{(\eta,\Lambda)} = \left\{\eta + \delta\eta, H + \frac{\partial \delta W}{\partial t}\right\}^{(\eta,\Lambda)}$$

Taking into account (2.21), (2.20), and (2.19), we obtain just equations (2.22).

If a physical quantity is represented by a function $F(\eta)$ in the variables η, then in the primed variables (2.10) it will be represented by a function $F'(\eta')$ which is related to the former one by $F'(\eta') = F(\eta)$. In the infinitesimal form this results in $F'(\eta) = F_{\delta W}(\eta)$, according to (2.22),

$$F_{\delta W}(\eta) = F(\eta) + \delta F(\eta), \qquad \delta F(\eta) = \{\delta W, F\}^{(\eta,\Lambda)} \tag{2.23}$$

Variations of the phase variables in course of the time evolution (2.20) can also be considered as a generalized canonical transformation. Namely, let η be the phase variables at a time instant t, and η_0 be those at the time instant $t = 0$. Then η are some function of η_0 and of t as a parameter, $\eta = \varphi(\eta_0, t)$. One can see that the transformation from η_0 to η is a generalized canonical transformation. Moreover, this transformation can be formally written explicitly. Indeed, considering for simplicity time-independent Hamiltonians only, one can see that the solution of the Cauchy problem for equation (2.20), with the initial data η_0 at $t = 0$, has the form

$$\eta = e^{\check{H}t}\eta_0 \tag{2.24}$$

where the operator $\check{H}$ is defined by its action on functions $F(\eta_0)$ of η_0 as $\check{H}F(\eta_0) = \{F(\eta_0), H(\eta_0)\}^{(\eta_0,\Lambda)}$. Because the transformation (2.24) is the generalized canonical transformation [see (2.10)] with the generating function $H(\eta_0)$, one has only to prove that (2.24) obeys the equation of motion (2.20). Taking the time derivative from (2.24), one gets

$$\dot{\eta} = \check{H}e^{\check{H}t}\eta_0 = \{e^{\check{H}t}\eta_0, H(\eta_0)\}^{(\eta_0,\Lambda)} \tag{2.25}$$

Using (2.12), one can verify that

$$H(e^{\check{H}t}\eta_0) = e^{\check{H}t}H(\eta_0) = H(\eta_0) \tag{2.26}$$

Substituting (2.26) into (2.25) and taking into account the property (2.19), one obtains

$$\dot{\eta} = \{e^{\check{H}t}\eta_0, H(e^{\check{H}t}\eta_0)\}^{(\eta_0,\Lambda)} = \{\eta, H(\eta)\}^{(\eta,\Lambda)}$$

which proves our claim.

3. D-TRANSFORMATIONS

Now we apply the previous consideration to theories with constraints, namely, with second-class constraints.

Let us consider a theory with second-class constraints $\Phi = (\Phi_l(\eta))$, in the Hamiltonian formulation, described by phase variables η^A, $A = 1, \ldots, 2n$, half of which are coordinates q and half are moments p, so that $\eta^A = (q^a, p_a)$, $A = (\zeta, a)$, $\zeta = 1, 2$, $a = 1, \ldots, n$. An important object in such theories is the Dirac bracket between two functions $F(\eta)$ and $G(\eta)$,

$$\{F, G\}_{D(\Phi)} = \{F, G\} - \{F, \Phi_l\}\{\Phi, \Phi\}^{-1}_{ll'}\{\Phi_{l'}, G\} \tag{3.1}$$

It is easy to see that the Dirac bracket is a particular case of the generalized Poisson bracket (2.1),

$$\{F, G\}_{D(\Phi)} = \{F, G\}^{(\eta,\Lambda)} \tag{3.2}$$

with

$$\Lambda^{AB} = E^{AB} - \{\eta^A, \Phi_l\}\{\Phi, \Phi\}^{-1}_{ll'}\{\Phi_{l'}, \eta^B\} = \{\eta^A, \eta^B\}_{D(\Phi)} \tag{3.3}$$

If so, then one can consider the generalized canonical transformations for such a generalized Poisson bracket. This special but important case of the generalized canonical transformations we will call D-transformations. Thus, by the definition, the D-transformation $\eta \to \eta'$ preserves the form invariance of the Dirac bracket,[2]

$$\{F, G\}_{D(\Phi)} = \{F', G'\}'_{D(\Phi)} \tag{3.4}$$

As we will see further, in theories with second-class constraints, D-transformations play the same role that canonical transformations play in theories without constraints.

An explicit form of D-transformations connected continuously with the identical transformation can be extracted from (2.10) and (3.2),

[2] A prime on the Dirac bracket in (3.4) means that the latter is calculated in the primed variables.

$$\eta' = e^{\check{W}}\eta, \qquad \check{W}F(\eta) = \{F, W\}_{D(\Phi)} \tag{3.5}$$

and in the infinitesimal form

$$\eta' = \eta + \delta\eta, \qquad \delta\eta = \{\eta, \delta W\}_{D(\Phi)} \tag{3.6}$$

where $W(\eta)$ is a generating function of the D-transformation.

One can see that D-transformations differ from canonical ones only by terms proportional to constraints. Indeed, the variation $\delta\eta$ under the D-transformation can be written as

$$\delta\eta = \{\eta, \delta W\}_{D(\Phi)} = \{\eta, \delta W'\} + \{\Phi\} \tag{3.7}$$

where

$$\delta W' = \delta W - \Phi_l\{\Phi, \Phi\}^{-1}_{ll'}\{\Phi_{l'}, \delta W\}$$

and $\{\Phi\}$ accumulates terms proportional to constraints, or terms which vanish on the constraint surface.

Equations of motion for a theory with second-class constraints can be written in the form (Dirac, 1964)

$$\dot{\eta} = \{\eta, H\}_{D(\Phi)} \tag{3.8}$$

$$\Phi(\eta) = 0 \tag{3.9}$$

They consist of two equations, a Hamiltonian equation (3.8) with the Dirac bracket, which is in the same time a generalized Poisson bracket, and the equation of constraints (3.9). Using the consideration of Section 2, one can say that equation (3.8) is form invariant under the D-transformations. It turns also out that the equation of constraints (3.9) is form invariant under the D-transformations. Indeed, let $\Phi'(\eta') = 0$ be the equation of constraints in variables η' connected to η by a D-transformation; then the relation

$$\Phi'(\eta') = \Phi(\eta) \tag{3.10}$$

has to hold. One can consider this as a functional equation for the function Φ'. It is easy to verify that they have a solution $\Phi' = \Phi$. Indeed, consider the function $\Phi(\eta')$. Using the formula (2.12) and a well-known property of the Dirac bracket, $\{F, \Phi_l\}_{D(\Phi)} = 0$, for any function $F(\eta)$ and any constraint Φ_l, we get

$$\Phi(\eta') = e^{\check{W}}\Phi(\eta) = \Phi(\eta) \tag{3.11}$$

This means that the constraints surface $\Phi(\eta) = 0$ after the D-transformation can be described by the same function, i.e., by the equation $\Phi(\eta') = 0$.

Thus, equations of motion of theories with second-class constraints are form invariant under the D-transformations. Namely, equations (3.8) and (3.9) have the following form after the D-transformation (3.6):

$$\dot{\eta}' = \{\eta', H'\}'_{D(\Phi)}, \qquad \Phi(\eta') = 0, \qquad H'(\eta') = H(\eta) + \frac{\partial \delta W}{\partial t} \tag{3.12}$$

or

$$\dot{\eta} = \left\{\eta, H_{\delta W} + \frac{\partial \delta W}{\partial t}\right\}_{D(\Phi)}, \qquad \Phi(\eta) = 0 \tag{3.13}$$

and the physical quantity F is described by the function $F_{\delta W}(\eta)$ [see (2.23)]

$$F'(\eta) = F_{\delta W}(\eta) = F(\eta) + \{\delta W, F\}_{D(\Phi)} \tag{3.14}$$

In the special canonical variables (ω, Ω) in which the equation of constraints has a simple form $\Omega = 0$ [Gitman and Tyutin, 1983, 1986, 1987, 1990) and the Dirac bracket reduces to the Poisson one in the variable ω, so that the latter is a physical variable on the constraints surface, D-transformations have a simple meaning: they are canonical transformations in the sector of physical variable ω with no change of variable Ω. This is natural because the D-transformations do not change the form of the constraints.

4. QUANTUM IMPLEMENTATION OF D-TRANSFORMATIONS

One can ask which kind of transformation in quantum theory corresponds to D-transformations in classical theory. It is easy to see that these are unitary transformations and vice versa: unitary transformations in a quantum theory with constraints induce in a sense D-transformations in the corresponding classical theory. From this point of view D-transformations in theories with constraints play a role similar to that of the canonical transformations in theories without constraints. To prove this, we have to remember that in a classical theory D-transformations are transformations of trajectories-states of the theory. Thus, to speak literally, some transformations of quantum states-vectors in a Hilbert space have to correspond to them in a quantum theory.

Let us have a classical theory with second-class constraints which is described by the equations of motion (3.8), (3.9). Its canonical quantization (Dirac, 1964; Gitman and Tyutin, 1986, 1990) consists formally in a transition from the classical variables η to the quantum operators $\hat{\eta}$, $P(\hat{\eta}^A) = P(\eta^A) = P_A$, which obeys the operator relations[3]

$$[\hat{\eta}^A, \hat{\eta}^B\} = i\hbar\widehat{\overline{\{\eta^A, \eta^B\}_{D(\Phi)}}} = i\hbar\widehat{\overline{\Lambda^{AB}(\eta)}}, \qquad \widehat{\overline{\Phi(\eta)}} = 0 \tag{4.1}$$

and which is supposed to be realized in a Hilbert space $\mathfrak{R}$ of vectors $|\Psi\rangle$. Then one has to assign the operator $\hat{F}$ to each physical quantity F which is described in the classical theory by the function $F(\eta)$, using a certain correspondence rule, $\hat{F} = \widehat{\overline{F(\eta)}}$. The time evolution of the state vectors is defined by the quantum Hamiltonian $\hat{H} = \widehat{\overline{H(\eta)}}$, according to the Schrödinger equation

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = \hat{H}|\Psi\rangle \tag{4.2}$$

Let us consider a unitary transformation of the state vectors, $|\Psi\rangle \to |\Psi'\rangle = \hat{U}|\psi\rangle$, where $\hat{U}$ is some unitary operator, $\hat{U}^+\hat{U} = 1$, which one can write in the form

$$\hat{U} = \exp\left\{-\frac{i}{\hbar}\hat{W}\right\} \tag{4.3}$$

where $\hat{W}$ is a Hermitian operator, $\hat{W}^+ = \hat{W}$, further called the quantum generator of the transformation. In the infinitesimal form ($\hat{W} \to \delta\hat{W}$), simplifying the consideration, we have $|\Psi'\rangle = |\Psi\rangle + \delta|\Psi\rangle$, $\delta|\Psi\rangle = -(i/\hbar)\delta\hat{W}|\Psi\rangle$.

One can find the variation of operators of physical quantities from the condition $\langle\Psi|\hat{F}|\Psi\rangle = \langle\Psi'|\hat{F}'|\Psi'\rangle$, which results in

$$\hat{F}' = \hat{F}_{\delta W} = \hat{U}\hat{F}\hat{U}^+ = \hat{F} + \delta\hat{F}, \qquad \delta\hat{F} = -\frac{i}{\hbar}[\delta\hat{W}, \hat{F}] \tag{4.4}$$

If $\delta W(\eta)$ is a symbol for the operator $\delta\hat{W}$, $\delta\hat{W} = \widehat{\overline{\delta W(\eta)}}$, and $F(\eta)$ is one of the operators $\hat{F}$ (the classical function which describes the physical quantity in the variables η), $\hat{F} = \widehat{\overline{F(\eta)}}$, then it follows from (4.1) that

$$\delta\hat{F} = \widehat{\overline{\{\delta W, F\}_{D(\Phi)}}} + o(\hbar) \tag{4.5}$$

Recalling formula (3.14), one can write

[3] By $[\hat{A}, \hat{B}\}$ we denote a generalized commutator of two operators $\hat{A}$ and $\hat{B}$, with definite parities $P(\hat{A})$ and $P(\hat{B})$, $[\hat{A}, \hat{B}\} = \hat{A}\hat{B} - (-1)^{P(\hat{A})P(\hat{B})}\hat{B}\hat{A}$. An overbar with a caret above a classical function $A(\eta)$ here and further means a certain rule of correspondence between the function and the corresponding quantum operator $\hat{A}$, $\hat{A} = \widehat{\overline{A(\eta)}}$. The former is in this case the symbol of the operator (Berezin, 1965, 1983, 1987). The choice of this rule is not important in our considerations.

$$\hat{F}_{\delta W} = \overline{F_{\delta W}(\eta)}^{\wedge} + o(\hbar) \tag{4.6}$$

Thus, operators of physical quantities transformed in the course of a unitary transformation have as their symbols initial classical functions transformed by a D-transformation, with the generating function being a classical symbol of the quantum generator of the unitary transformation.

The Schrödinger equation for transformed vectors can be derived from (4.2) and has the form

$$i\hbar\,\frac{\partial|\Psi'\rangle}{\partial t} = \hat{H}'|\Psi'\rangle, \qquad \hat{H}' = \hat{H}_{\delta W} + \frac{\partial}{\partial t}\,\overline{\delta W}^{\wedge} \tag{4.7}$$

Thus, the time evolution of the state vectors after the unitary transformation is governed by a quantum Hamiltonian with the classical symbol

$$H'(\eta) = H_{\delta W}(\eta) + \frac{\partial \delta W(\eta)}{\partial t} + o(\hbar) \tag{4.8}$$

That fact and (4.1) allow one to see that the classical limit of the quantum theory after the unitary transformation (4.3) is described by equations (3.13) and therefore corresponds to the D-transformed classical theory with a generating function which is a classical symbol of the quantum generator of the unitary transformation. In the same way one can prove the inverse statement: if we have a classical theory and its D-transformed formulation, then quantum versions of both theories are connected by a unitary transformation. In addition, the classical generating function of the D-transformation and the quantum generator of the unitary transformation are connected in the above manner.

Consider now the generating functional $Z(J)$ of Green's functions for a theory with second-class constraints in the form of a Hamiltonian path integral and the behavior of the latter under the D-transformations. Such an integral can be written in the form

$$Z(J) = \int \exp\left\{\frac{i}{\hbar}\,S_J(\eta)\right\}\mathcal{D}\eta \tag{4.9}$$

where

$$S_J(\eta) = \int [p_a\dot{q}^a - H_J(\eta)]\,dt, \qquad H_J(\eta) = H(\eta) + J_A\eta^A$$

is the classical action with sources, $J_A(t)$ are sources to the variables $\eta^A(t)$, $P(J_A) = P(\eta^A) = P_A$, and the measure $\mathcal{D}\eta$ has the form (Faddeev, 1969; Fradkin, 1973)

$$\mathcal{D}\eta = \text{Sdet}^{1/2}\{\Phi, \Phi\}\delta(\Phi)\, D\eta \tag{4.10}$$

with Sdet$\{\Phi, \Phi\}$ denoting the superdeterminant of the matrix $\{\Phi_l, \Phi_m\}$.

As is known, if a change of variables $\eta' = \eta'(\eta)$ is a canonical transformation, then $|\text{Ber}\ \eta'(\eta)| = 1$, where Ber $\eta'(\eta)$ is the Berezinian (Berezin, 1965, 1983, 1987) of the change of variables, Ber $\eta'(\eta) = \text{Sdet}\ \partial_r\eta'^A/\partial\eta^B$. In particular, for infinitesimal canonical transformations $\eta' = \eta + \delta\eta$, $\delta\eta = \{\eta, \delta W\}$, Ber $\eta'(\eta) = 1$. In the case of theories without constraints, the measure $\mathcal{D}\eta$, (4.10), reduces to $D\eta$ and is invariant under canonical transformations, but in theories with constraints it is not. However, this measure is invariant under D-transformations,

$$\mathcal{D}\eta' = \mathcal{D}\eta$$

which confirms once again that the latter play the role of canonical transformations in theories with constraints. To see this one can use the relation (Gitman and Tyutin, 1983, 1987)

$$\text{Sdet}^{1/2}\{\Phi, \Phi\}\delta(\Phi)\big|_{\eta\to\eta'(\eta)}\ \text{Ber}\ \eta'(\eta) = \text{Sdet}^{1/2}\{\Phi, \Phi\}\delta(\Phi) \tag{4.11}$$

where $\eta' = \eta + \{\eta, \delta W\}_{D(\Phi)}$.

The invariance of the measure (4.10) under D-transformations induces an invariance of the integral (4.9) under the transformation of the action $S_J(\eta)$,

$$S_J(\eta) \to S'_J(\eta) = S_J(\eta'(\eta)) = S_J(\eta) + \delta S_J(\eta) \tag{4.12}$$

where $\eta'(\eta)$ is a D-transformation,

$$Z(J) = \int \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = \int \exp\left\{\frac{i}{\hbar} S'_J(\eta)\right\} \mathcal{D}\eta$$

or

$$\int \delta S_J(\eta) \exp\left\{\frac{i}{\hbar} S_J(\eta)\right\} \mathcal{D}\eta = 0 \tag{4.13}$$

It is enough to know $\delta S_J(\eta)$ on the constraints surface, because the integration in (4.13) only goes over this surface due to the δ-function in the measure (4.10). Taking into account the representation (3.7), one can find an expression for $\delta S_J(\eta)$ on the constraints surface,

$$\delta S_J(\eta)\big|_{\Phi=0} = (p\delta q - \delta W)\big|_{t_{\text{in}}}^{t_{\text{out}}} + \int \left[\frac{\partial}{\partial t}\delta W - \{H_J, \delta W\}_{D(\Phi)}\right] dt \tag{4.14}$$

In field theory usually $t_{\text{in,out}} \to \pm\infty$ and trajectories of integration vanish at these time limits. Considering D-transformations, which do not change this property, one gets

$$\int \left[\int \left(\frac{\partial}{\partial t} \delta W - \{H_J, \delta W\}_{D(\Phi)} \right) dt \right] \exp\left\{ \frac{i}{\hbar} S_J(\eta) \right\} \mathcal{D}\eta = 0 \quad (4.15)$$

This relation can be used to obtain different kinds of equations for the generating functional and therefore for Green's functions. For example, let us consider D-transformations with two types of generating functions: $\delta W = \epsilon_A \eta^A$ and $\delta W = \zeta_l \Phi_l(\eta)$, with arbitrary "small" time-dependent functions $\epsilon_A(t)$ and $\zeta_l(t)$. Using these δW in (4.15), we get two relations,

$$\int [\dot{\eta}^A - \{\eta^A, H_J\}_{D(\Phi)}] \exp\left\{ \frac{i}{\hbar} S_J(\eta) \right\} \mathcal{D}\eta = 0$$

$$\int \Phi_l(\eta) \exp\left\{ \frac{i}{\hbar} S_J(\eta) \right\} \mathcal{D}\eta = 0 \quad (4.16)$$

which can be rewritten in the form of Schwinger equations for the functional $Z(J)$,

$$[\dot{\eta}^A - \{\eta^A, H_J\}_{D(\Phi)}]_{\eta \to \delta_l/\delta(iJ)} Z(J) = 0, \qquad \Phi\left(\frac{\delta_l}{\delta(iJ)} \right) Z(J) = 0 \quad (4.17)$$

5. REMARKS

We have demonstrated that in theories with second-class constraints D-transformations play the usual role of canonical transformations. In fact, in Gitman and Tyutin (1986, 1990) we already used infinitesimal D-transformations for technical reasons, but at that time we did not fully realize their special role.

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